Put ~~...~~

~~...~~

(1) $\int x^3 e^{x^2} dx = \int x^2 e^{x^2} \cdot x\,dx$

Put $x^2 = t \Rightarrow 2x\,dx = dt$

$\Rightarrow I = \frac{1}{2}\int t\, e^{t}\, dt$

$I = \frac{1}{2}\left[t\int e^t\,dt - \int \frac{d}{dt}(t)\left(\int e^t\,dt\right)dt \right]$

$= \frac{1}{2}\left(te^t - \int 1\cdot e^t\,dt\right)$

$= \frac{1}{2}\left(te^t - e^t\right) + C$

$\Rightarrow \frac{e^t}{2}(t-1) + C$

$\Rightarrow \frac{e^{x^2}}{2}(x^2-1) + C$

(2) $I = \int x^3 e^{2x}\,dx$

$\Rightarrow \frac{x^3 e^{2x}}{2} - \int 3x^2 \frac{e^{2x}}{2}\,dx = \frac{x^3 e^{2x}}{2} - \frac{3}{2}\int x^2 e^{2x}\,dx$

$\Rightarrow \frac{x^3 e^{2x}}{2} - \frac{3}{2}\left\{ \frac{x^2 e^{2x}}{2} - \int 2x \cdot \frac{e^{2x}}{2}\,dx \right\}$

$\Rightarrow \frac{x^3 e^{2x}}{2} - \frac{3}{4}x^2 e^{2x} + \frac{3}{2}\int x e^{2x}\,dx$

$\Rightarrow \frac{x^3 e^{2x}}{2} - \frac{3}{4}x^2 e^{2x} + \frac{3}{2}\left(\frac{x e^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx\right)$

$\Rightarrow \frac{x^3 e^{2x}}{2} - \frac{3}{4}x^2 e^{2x} + \frac{3}{4}x e^{2x} - \frac{3}{8}e^{2x}$

$I = \int \sec^3 x \, dx = \int \sec x \cdot \sec^2 x \, dx$

(1st: $\sec x$, 2nd: $\sec^2 x$)

$I = \sec x \tan x - \int \sec x \tan x \tan x \, dx$

$= \sec x \tan x - \int \sec x \tan^2 x \, dx.$

$= \sec x \tan x - \int (\sec^3 x - \sec x) \, dx$

$= \sec x \tan x - \underline{\int \sec^3 x \, dx} + \int \sec x \, dx.$

$= \sec x \tan x - I + \log(\sec x + \tan x)$

$\Rightarrow 2I = \sec x \tan x + \log(\sec x + \tan x)$

$\Rightarrow I = \frac{1}{2} \sec x \tan x + \frac{1}{2} \log(\sec x + \tan x)$

$I = \int \log x \, dx = \int 1 \cdot \log x \, dx.$

(2nd: 1, 1st: $\log x$)

$= \log x \cdot (x) - \int \frac{1}{x} \cdot x \, dx$

$= x \log x - x \Rightarrow \log x^x - x$

$\Rightarrow x(\log x - 1)$

$I = \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} \, dx$

Put $\sin^{-1} x = t \Rightarrow x = \sin t$

$\frac{1}{\sqrt{1-x^2}} dx = dt$

$I = \int (\sin t) \, t \, dt$

$\Rightarrow I = t(-\cos t) - \int 1(-\cos t)dt$

$= -t\cos t + \sin t$

$= -\sin^{-1}x(\cos\sin^{-1}x) + \sin(\sin^{-1}x) + C$

$= -\sin^{-1}x(\cos(\sin^{-1}x)) + x + C$

$$\boxed{I = \int e^x(f(x) + f'(x))dx = e^x f(x)}$$

① $I = \int e^x(\sin x + \cos x)dx.$

$= e^x \sin x.$

② $\int \frac{e^x(1 + x\log x)}{x} dx$

$\Rightarrow \int e^x\left(\frac{1}{x} + \log x\right)dx = e^x \log x.$

③ $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx.$

$= \int e^x\left(\frac{1 + 2\sin x/2 \cos x/2}{2\cos^2 x/2}\right)dx$

$= \int e^x\left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right)dx$

$\frac{1}{2}\sec^2\frac{x}{2} \to f'(x)$, $\tan\frac{x}{2} \to f(x)$

$= e^x \tan\frac{x}{2} + C.$

Ex.6. *Evaluate* $\int \frac{e^x(1-x)^2}{(1+x^2)^2}dx.$ [P.U. 60A; R.U. 92H]

Soln. Let the given integral be I.

Then $$I = \int \frac{e^x\{(1+x^2)-2x\}}{(1+x^2)^2}dx$$

$$= \int e^x \left\{\frac{1}{1+x^2} - \frac{2x}{(1+x^2)^2}\right\}dx.$$

Let $f(x) = \frac{1}{1+x^2}$ so that $f'(x) = -\frac{2x}{(1+x^2)^2}$.

Therefore $$I = \int e^x\{f(x) + f'(x)\}dx = e^x f(x)$$

$$= \frac{e^x}{1+x^2}.$$

Ex.7. *Evaluate* $\int \left\{\frac{1}{\log x} - \frac{1}{(\log x)^2}\right\}dx.$

[B.U. 57A; R.U. 65S; Bhag. 63S; M.U. 77A]

Soln. Let the given integral be I.

Put $\log x = y$ so that $x = e^y$ and therefore $dx = e^y\, dy$.

Then $$I = \int \left\{\frac{1}{y} - \frac{1}{y^2}\right\}e^y dy$$

Let $f(y) = \frac{1}{y}$ so that $f'(y) = -\frac{1}{y^2}$.

Therefore $$I = \int e^y\{f(y) + f'(y)\}dy$$

$$= e^y f(y) = \frac{e^y}{y}$$

$$= \frac{e^{\log x}}{\log x}$$ after making the substitution $y = \log x$

$$= \frac{x}{\log x}.$$

Ex.3. **Evaluate** $\int \frac{e^{m\tan^{-1}x}}{(1+x^2)^2}dx.$

[P.U. 51S, 52S, 79A; B.U. 53A, 62S; Bhag. 65S, 66S]

Soln. Let $I = \int \frac{e^{m\tan^{-1}x}}{(1+x^2)^2}dx.$

We put $\tan^{-1}x = \theta$ so that $x = \tan\theta$ and therefore

$$dx = \sec^2\theta\, d\theta.$$

$$\therefore \quad I = \int \frac{e^{m\theta}\sec^2\theta\, d\theta}{(1+\tan^2\theta)^2} = \int \frac{e^{m\theta}\sec^2\theta d\theta}{\sec^4\theta}$$

$$= \int e^{m\theta}.\frac{1}{\sec^2\theta}d\theta = \int e^{m\theta}\cos^2\theta d\theta$$

$$= \int e^{m\theta}.\frac{1+\cos 2\theta}{2}d\theta = \frac{1}{2}\int e^{m\theta}(1+\cos 2\theta)d\theta$$

$$= \frac{1}{2}\left[\int e^{m\theta}d\theta + \int e^{m\theta}\cos 2\theta\, d\theta\right]$$

$$= \frac{1}{2}\cdot\frac{e^{m\theta}}{m} + \frac{1}{2}\cdot\frac{e^{m\theta}}{m^2+4}(m\cos 2\theta + 2\sin 2\theta)$$

$$= \frac{e^{m\theta}}{2m} + \frac{1}{2}\cdot\frac{e^{m\theta}}{4+m^2}\left\{m\cdot\frac{1-\tan^2\theta}{1+\tan^2\theta} + 2\cdot\frac{2\tan\theta}{1+\tan^2\theta}\right\}$$

$$= \frac{e^{m\theta}}{2m} + \frac{1}{2}\cdot\frac{e^{m\theta}}{4+m^2}\left\{m\frac{1-x^2}{1+x^2} + 2\cdot\frac{2x}{1+x^2}\right\}$$

$$= \frac{e^{m\tan^{-1}x}}{2m} + \frac{1}{2}\frac{e^{m\tan^{-1}x}}{4+m^2}\left\{\frac{m(1-x^2)+4x}{1+x^2}\right\}$$

$$= \frac{1}{2}\cdot e^{m\tan^{-1}x}\left[\frac{1}{m} + \frac{1}{4+m^2}\left\{\frac{m(1-x^2)+4x}{1+x^2}\right\}\right]$$

EXAMPLE 3

Evaluate

1. (i) $\int x\cos x\, dx$ (ii) $\int x\sin 3x\, dx$

   (iii) $\int x^2\cos x\, dx$ (iv) $\int x\sin x\cos x\, dx$

(v) $\int x \sin x \sin 2x \, dx$

(vi) $\int x \cos 2x \cos 3x \, dx$

(vii) $\int x \sin^2 x \, dx$

(viii) $\int x \cos^3 x \, dx$

(ix) $\int x \sin^3 x \, dx$

(x) $\int x^2 \sin^2 x \, dx$

(xi) $\int x^2 \cos^2 x \, dx$

(xii) $\int x \sin^2 x \cos^2 x \, dx$

2. (i) $\int x^2 e^{2x} \, dx$

(ii) $\int x \log x \, dx$

(iii) $\int \frac{\log x}{x} \, dx$

(iv) $\int x^2 \log x \, dx$

3. $\int 2^x \cdot x \, dx$ [P.U. 65A]

4. $\int x \sec^2 x \, dx$ [B.U. 68A]

5. $\int \frac{x}{1 + \cos x} dx$ [M.U. 68A]

6. $\int \operatorname{cosec}^3 x \, dx$

7. (i) $\int e^x \sin x \, dx$

(ii) $\int e^x \cos x \, dx$

(iii) $\int e^x \sin x \cos x \, dx$

(iv) $\int e^{-2x} \sin 2x dx$

(v) $\int e^x \sin^2 x \, dx$

(vi) $\int e^x \cos^2 x \, dx$

(vii) $\int e^x \sin x \sin 2x \, dx$

(viii) $\int 2^x \sin x \, dx$

(ix) $\int e^{\sin x} \sin 2x \, dx$

(x) $\int \frac{\cos^2 x}{e^{2x}} dx$

(xi) $\int \frac{\cos^3 x}{e^{3x}} dx$ [Bhag. 65A]

8. $\int \frac{e^{m \tan^{-1} x}}{(1 + x^2)^{3/2}} dx$ [P.U. 46A]

9. If $\upsilon = \int e^{ax} \sin bx \, dx$, $u = \int e^{ax} \cos bx \, dx$, prove that

(i) $\tan^{-1}\left(\frac{\upsilon}{u}\right) + \tan^{-1}\left(\frac{b}{a}\right) = bx$

[P.U. 68S; B.U. 60A, 62A; R.U. 69A; Bhag. 63]

(ii) $(a^2 + b^2)(u^2 + \upsilon^2) = e^{2ax}$.

[**Soln.** From the formula,

$$\upsilon = \int e^{ax} \sin bx \, dx$$

$$= \frac{1}{\sqrt{a^2 + b^2}} e^{ax} \sin\left(bx - \tan^{-1}\frac{b}{a}\right) \qquad ...(1)$$

and $u = \int e^{ax} \cos bx \, dx$

$$= \frac{1}{\sqrt{a^2+b^2}} e^{ax} \cos\left(bx - \tan^{-1}\frac{b}{a}\right) \quad \cdots(2)$$

On division, $\frac{v}{u} = \tan\left(bx - \tan^{-1}\frac{b}{a}\right)$

$\Rightarrow \quad \tan^{-1}\frac{v}{u} = bx - \tan^{-1}\frac{b}{a}$

$\Rightarrow \quad \tan^{-1}\frac{v}{u} + \tan^{-1}\frac{b}{a} = bx$

Squaring (1) and (2) and adding we get (ii).]

10. $\int \tan^{-1}x \, dx$ [P.U. 23A; M.U. 69S]

11. $\int \log(x^2 + a^2)dx$ [P.U. 53S]

12. $\int \log\{x + \sqrt{a^2+x^2}\}dx$ [B.U. 64A; R.U. 63A]

13. $\int \frac{\tan^{-1}x}{(1+x^2)^{3/2}}dx$ [P.U. 48S]

14. $\int \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}dx$ [P.U. 37S, 46S; M.U. 70S]

15. $\int \frac{x^2\tan^{-1}x}{1+x^2}dx$

16. $\int x^2 \sin^{-1}x \, dx$ [P.U. 30A; M.U. 78A]

17. $\int e^x(1 + \tan x)\sec x \, dx$ [P.U. 51A]

18. $\int e^x(\sin x + \cos x)\sec^2 x \, dx$ 19. $\int e^x(\tan x - \log\cos x)dx$

20. $\int xe^x \sin^2 x \, dx$ [R.U. 62A]

[Hint : $I = \int xe^x\left(\frac{1-\cos 2x}{2}\right)dx$

$$= \frac{1}{2}\int xe^x dx - \frac{1}{2}\int x(e^x \cos 2x)dx = I_1 - I_2.$$

But $I_2 = \frac{1}{2}\int x(e^x \cos 2x)dx$

$$= \frac{1}{2}[x\int e^x \cos 2x dx - \int 1 \cdot \{\int e^x \cos 2x \, dx\}dx.$$

Now substitute $\int e^x \cos 2x \, dx = \frac{1}{\sqrt{5}}\cos(2x - \tan^{-1}2)$]

21. $\int x^2 e^x \sin x \, dx$ [M.U. 68A]

22. $\int \log x \cdot \sin^{-1}x \, dx$

23. $\int \log \frac{a}{x} \sin^{-1}x \, dx$ [P.U. 59A]

[The given integral $= \int (\log a - \log x)\sin^{-1}x \, dx$

$= \log a \int \sin^{-1}x \, dx - \int \log x \sin^{-1}x \, dx$

Then put $\sin^{-1}x = \theta$.]

24. $\int \frac{x^2 dx}{(x \sin x + \cos x)^2}$

[P.U. 55H, 56H; M.U. 63H; R.U. 72A; P.U. 66A, 77A; M.U. 78A; B.U. 93H]

[**Soln. :** Out starting point is the differentiation of

$$\frac{1}{x \sin x + \cos x}.$$

Since $\frac{d}{dx}\left\{\frac{1}{x \sin x + \cos x}\right\}$

$$= \frac{(x \sin x + \cos x)\ (\text{d.c. of } 1) - 1 \cdot \text{d.c. } (x \sin x + \cos x)}{(x \sin x + \cos x)^2}$$

$$= \frac{(x \sin x + \cos x) \times 0 - (x \cos x + \sin x - \sin x)}{(x \sin x + \cos x)^2}$$

$$= \frac{-x \cos x}{(x \sin x + \cos x)^2}$$

$$\therefore \int \frac{x \cos x}{(x \sin x + \cos x)^2} dx = -\frac{1}{x \sin x + \cos x}.$$

Now $I = \int \frac{x^2 dx}{(x \sin x + \cos x)^2}$

$$= \int \frac{x}{\cos x} \cdot \frac{x \cos x}{(x \sin x + \cos x)^2} dx$$

which by using integration by psrts, taking $\frac{x}{\cos x}$ as the first function,

$$= \frac{x}{\cos x}\left\{-\frac{1}{x \sin x + \cos x}\right\} - \int \left\{\text{d.c. of } \frac{x}{\cos x}\right\}\left\{-\frac{1}{x \sin x + \cos x}\right\} dx$$

$$= -\frac{x}{\cos x(x \sin x + \cos x)} + \int \frac{\cos x + x \sin x}{\cos^2 x} \times \frac{1}{x \sin x + \cos x} dx$$

$$= -\frac{x}{\cos x(x \sin x + \cos x)} + \int \sec^2 x \, dx$$

$$= -\frac{x}{\cos x \, (x \sin x + \cos x)} + \tan x$$

$$= \frac{\sin x}{\cos x} - \frac{x}{\cos x(x \sin x + \cos x)}$$

$$= \frac{x \sin^2 x + \sin x \cos x - x}{\cos x(x \sin x + \cos x)}$$

$$= \frac{\sin x \cos x - x(1 - \sin^2 x)}{\cos x(x \sin x + \cos x)} = \frac{\sin x \cos x - x \cos^2 x}{\cos x(x \sin x + \cos x)}$$

$$= \frac{\sin x - x \cos x}{x \sin x + \cos x}$$

25. $\int \frac{x^2 dx}{(x \cos x - \sin x)^2}$ [P.U. 57H]

26. $\int \operatorname{cosec}^2 x \log (\cos x + \sqrt{\cos 2x}) dx$ [P.U. 46H; M.U. 65H]

[**Hint :** Put $\cot x = z$, $\therefore \tan x = \frac{1}{z}$ and hence $\cos x = \frac{z}{\sqrt{z^2 + 1}}$

and $\cos 2x = 2 \cos^2 x - 1 = \frac{z^2 - 1}{z^2 + 1}$.

Now integrate by parts, taking $\log (\cos x + \sqrt{\cos 2x})$ as the first function and $\operatorname{cosec}^2 x$ as the second.]

ANSWERS

1. (i) $x \sin x + \cos x$ (ii) $\frac{1}{9} \sin 3x - \frac{x \cos 3x}{3}$

(iii) $x^2 \sin x + 2x \cos x - 2 \sin x$ (iv) $-\frac{x \cos 2x}{4} + \frac{\sin 2x}{8}$

(v) $\frac{1}{2}\left[x \sin x - \frac{x \sin 3x}{3} + \cos x - \frac{\cos 3x}{9}\right]$

(vi) $\frac{1}{2}\left[\frac{x \sin 5x}{5} + \frac{\cos 5x}{25} + x \sin x + x \cos x\right]$

(vii) $\frac{1}{4}\left[x^2 - \frac{1}{2} \cos 2x - x \sin 2x\right]$

(viii) $\frac{1}{4}\left[3(x \sin x + \cos x) + \frac{1}{3} x \sin 3x + \frac{1}{9} \cos 3x\right]$

(ix) $\frac{1}{4}\left[3(\sin x - x\cos x) + \frac{1}{3}x\cos 3x - \frac{1}{9}\sin 3x\right]$

(x) $\frac{1}{2}\left[\frac{x^3}{3} - \frac{x^2\sin 2x}{2} - \frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]$

(xi) $\frac{1}{2}\left[\frac{x^3}{3} + \frac{x^2\sin 2x}{2} + \frac{x\cos 2x}{2} - \frac{\sin 2x}{4}\right]$

(xii) $\frac{1}{16}\left[x^2 - \frac{x\sin 4x}{2} - \frac{\cos 4x}{8}\right]$

2. (i) $\frac{e^{2x}}{2}\left[x^2 - x + \frac{1}{2}\right]$ (ii) $\frac{x^2}{4}(2\log x - 1)$

(iii) $\frac{1}{2}(\log x)^2$ (iv) $\frac{x^3}{27}(9\log x - 1)$

3. $2^x(x\log 2 - 1)(\log 2)^2$ 4. $x\tan x + \log\cos x$

5. $x\tan\frac{x}{2} + 2\log\cos\frac{x}{2}$ 6. $\frac{1}{2}\log\tan\frac{x}{2} - \frac{1}{2}\operatorname{cosec} x\cot x$

7. (i) $\frac{1}{2}e^x(\sin x - \cos x)$ (ii) $\frac{1}{2}e^x(\sin x + \cos x)$

(iii) $\frac{1}{10}e^x(\sin 2x - 2\cos 2x)$ (iv) $-\frac{1}{4}e^{-2x}(\sin 2x + \cos 2x)$

(v) $\frac{1}{2}e^x - \frac{1}{10}e^x(\cos 2x + 2\sin 2x)$

(vi) $\frac{1}{2}e^x + \frac{1}{10}e^x(\cos 2x + 2\sin 2x)$

(vii) $\frac{1}{4}e^x[(\cos x + \sin x) - \frac{1}{5}(\cos 3x + 3\sin 3x)]$

(viii) $2^x(\log 2\sin x - \cos x)/\{1 + (\log 2)^2\}$

(ix) $2(\sin x - 1)e^{\sin x}$

(x) $\frac{1}{4}e^{-2x}\left[-1 + \frac{1}{2}(\sin 2x - \cos 2x)\right]$

(xi) $\frac{1}{8}e^{-3x}\left\{\frac{1}{3}(\sin 3x - \cos 3x) + \frac{3}{5}(\sin x - 3\cos x)\right\}$

8. $\frac{e^{m\tan^{-1}x}}{\sqrt{1+m^2}}\cos\left(\tan^{-1}x - \tan^{-1}\frac{1}{m}\right)$

10. $x\tan^{-1}x - \frac{1}{4}\log(1+x^2)$

11. $x\log(x^2+a^2) - 2x + 2a\tan^{-1}\frac{x}{a}$

12. $x\log(x + \sqrt{x^2+a^2}) - \sqrt{a^2+x^2}$

13. $\theta \sin \theta + \cos \theta$ where $\tan \theta = x$

14. $\dfrac{x}{\sqrt{1+x^2}} - \dfrac{1}{\sqrt{1+x^2}} \tan^{-1}x$

15. $x \tan^{-1}x - \dfrac{1}{2} \log (1 + x^2) - \dfrac{(\tan^{-1}x)^2}{2}$

16. $\dfrac{x^3}{3} \sin^{-1}x + \dfrac{\sqrt{1-x^2}}{3} - \dfrac{(1-x^2)^{3/2}}{9}$ 17. $e^x \sec x$ 18. $e^x \sec x$

19. $-e^x \log \cos x$

20. $\dfrac{1}{2}(x-1)e^x - \dfrac{1}{2\sqrt{5}} xe^x \cdot \cos (2x - \tan^{-1}2) - \dfrac{1}{10} e^x \cos(2x - \tan^{-1}2)$

21. $\dfrac{1}{\sqrt{2}} x^2 e^x \sin \left(x - \dfrac{\pi}{4}\right) + xe^x \cos x - \dfrac{1}{\sqrt{2}} e^x \sin \left(x - \dfrac{\pi}{4}\right)$

22. $-x \sin^{-1}x - \sqrt{1-x^2} \log \left(\dfrac{e}{x}\right) - \sqrt{1-x^2} + \cos h^{-1}\left(\dfrac{1}{x}\right)$

23. $x \sin^{-1}x + \sqrt{1-x^2} \log \dfrac{ae}{x} + \sqrt{1-x^2} - \cos h^{-1}\left(\dfrac{1}{x}\right)$

24. $\dfrac{\sin x - x \cos x}{x \sin x + \cos x}$ 25. $\dfrac{x \sin x + \cos x}{x \cos x - \sin x}$

26. $-\cot x \log (\cos x + \sqrt{\cos 2x}) + \text{cosec } x\sqrt{\cos 2x} - \cot x - x.$

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