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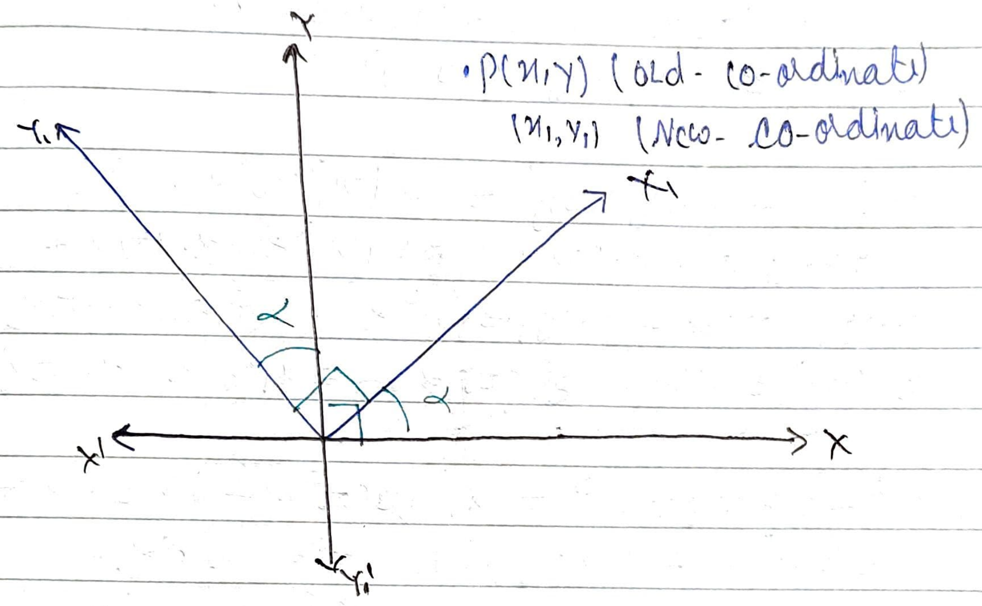
INVARIANTS :

THEOREM:

If by change of axes without change of origin the expression
$ax^2 + 2hxy + by^2$ becomes $a_1x_1^2 + 2h_1x_1y_1 + b_1y_1^2$
To prove that
i) $a+b = a_1+b_1$
2) $ab - h^2 = a_1b_1 - h_1^2$
3) $(a-b)^2 + 4h^2 = (a_1-b_1)^2 + 4h_1^2$

Proof:



given expression
$ax^2 + 2hxy + by^2$ ——— ①

Let us take O as origin and OX, OY as co-ordinate axes. Suppose the axes be rotated through an angle about O. The new position of axes becomes OX_1, OY_1.
Let P be any pt. in the plane.

Let (x, y) and (x_1, y_1) be the coordinates of same point P referred to OX, OY and OX_1, OY_1 as coordinates axes.

then we have

$x = x_1 \cos\alpha - y_1 \sin\alpha$

$y = x_1 \sin\alpha + y_1 \cos\alpha$.

putting the value of x and y in eq^n (1) we get

$ax^2 + 2hxy + by^2 = a(x_1\cos\alpha - y_1\sin\alpha)^2 + 2h(x_1\cos\alpha - y_1\sin\alpha)(x_1\sin\alpha + y_1\cos\alpha) + b(x_1\sin\alpha + y_1\cos\alpha)^2$

or, $ax^2 + 2hxy + by^2 = a(x_1^2\cos^2\alpha + y_1^2\sin^2\alpha - 2x_1y_1\sin\alpha.\cos\alpha) + 2h(x_1^2\sin\alpha.\cos\alpha + x_1y_1\cos^2\alpha - x_1y_1\sin^2\alpha - y_1^2\sin\alpha.\cos\alpha) + b(x_1^2\sin^2\alpha + y_1^2\cos^2\alpha + 2x_1y_1\sin\alpha.\cos\alpha)$

or, $ax^2 + 2hxy + by^2 = x_1^2(a\cos^2\alpha + 2h\sin\alpha\cos\alpha + b\sin^2\alpha) + 2\{h(\cos^2\alpha - \sin^2\alpha) - a\sin\alpha.\cos\alpha + b\sin\alpha.\cos\alpha\}x_1y_1 + (a\sin^2\alpha - 2h\sin\alpha.\cos\alpha + b\cos^2\alpha)y_1^2$ —— (1)

But given that, by these transformation the expression

$ax^2 + 2hxy + by^2$ be reduces to

$a_1x_1^2 + 2hx_1y_1 + by_1^2$ —— (3).

the eq^n (2) and (3) must be identical on compairing we get

$a_1 = a\cos^2\alpha + 2h\sin\alpha\cos\alpha + b\sin^2\alpha$ — (iv).

$h_1 = h(\cos^2\alpha - \sin^2\alpha) - a\sin\alpha.\cos\alpha + b\sin\alpha\cos\alpha$ — (5).

$b_1 = a\sin^2\alpha - 2h\sin\alpha.\cos\alpha + b\cos^2\alpha$ — (6).

Proof of (i).

Adding (iv) and (6) we get.

$a_1 + b_1 = a(\cos^2\alpha + \sin^2\alpha) + 2h\sin\alpha.\cos\alpha + b(\sin^2\alpha + \cos^2\alpha) - 2h\sin\alpha\cos\alpha$

$\therefore \boxed{a + b = a_1 + b_1}$ — (*).

Proof of (iii).

we have

$(a_1 - b_1)^2 + 4h_1^2 = (a_1 - b_1)^2 + (2h_1)^2.$

or, $\{(a\cos^2\alpha + 2h\sin\alpha.\cos\alpha + b\sin^2\alpha) - (a\sin^2\alpha - 2h\sin\alpha.\cos\alpha + b\cos^2\alpha)\}^2 + \{2h(\cos^2\alpha - \sin^2\alpha) - a2\sin\alpha.\cos\alpha + b2\sin\alpha\cos\alpha\}^2.$

or, $\{a\cos^2\alpha + h\sin2\alpha + b\sin^2\alpha - a\sin^2\alpha + h\sin2\alpha - b\cos^2\alpha\}^2 + \{2h\cos2\alpha - a\sin2\alpha + b\sin2\alpha\}^2.$

or, $\{a(\cos^2\alpha - \sin^2\alpha) + 2h\sin2\alpha - b(\cos^2\alpha - \sin^2\alpha)\}^2 + \{2h\cos2\alpha - (a-b)\sin2\alpha\}^2$

or, $(a\cos2\alpha + 2h\sin2\alpha - b\cos2\alpha)^2 + \{2h\cos2\alpha - (a-b)\sin2\alpha\}^2$

$= \{(a-b)\cos 2\alpha + 2h \sin 2\alpha\}^2 + \{2h \cos 2\alpha$
$- (a-b) \sin 2\alpha\}^2$

$= (a-b)^2 \{\cos^2 2\alpha + 2(a-b)\cos 2\alpha \cdot 2h \sin 2\alpha$
$+ 4h^2 \sin^2 2\alpha + 4h^2 \cos^2 2\alpha - 2 \cdot 2h \cos 2\alpha$
$(a-b) \sin 2\alpha + (a-b)^2 \sin^2 2\alpha$

$= (a-b)^2 \{\cos^2 2\alpha + \sin^2 2\alpha) + 4h^2 (\sin^2 2\alpha + \cos^2$

$= (a-b)^2 + 4h^2.$

$\therefore \boxed{(a_1 - b_1)^2 + 4h_1^2 = (a-b)^2 + 4h^2}$ —— (*)(*)

from (*) (*)

$(a_1 - b_1)^2 + 4h_1^2 = (a-b)^2 + 4h^2.$

$(a_1 + b_1)^2 - 4a_1 b_1 + 4h_1^2 = (a+b)^2 - 4ab + 4h^2.$

$\cancel{(a+b)^2} - 4(a_1 b_1 - h_1^2) = \cancel{(a+b)^2} - 4(ab - h^2).$

from (*) $a_1 - b_1 = a - b.$

or, $-4(a_1 b_1 - h_1^2) = -4(ab - h^2).$

$\therefore \boxed{a_1 b_1 - h_1^2 = ab - h^2}.$